

# NORM DECREASING HOMOMORPHISMS OF MEASURE ALGEBRAS

BY  
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**1. Introduction.** Let  $G$  be a locally compact group (=locally compact Hausdorff topological group). By the measure algebra of  $G$  we mean the Banach  $*$ -algebra  $M(G)$  of bounded regular Borel measures on  $G$ . It is well known that  $M(G)$  is the dual of  $C_0(G)$  the Banach space of continuous complex-valued functions on  $G$  which "vanish at infinity". The group algebra  $L^1(G)$  is the  $*$ -subalgebra of  $M(G)$  consisting of all measures that are absolutely continuous with respect to the Haar measure on  $G$ . Alternatively  $L^1(G)$  can be defined as the Banach  $*$ -algebra of (equivalence classes of) Haar summable functions on  $G$ .

Let  $F$  and  $G$  be locally compact groups,  $\alpha$  a bicontinuous isomorphism of  $F$  onto  $G$ , and  $\gamma$  a character on  $F$ . For  $\mu$  in  $M(F)$  and  $f$  in  $C_0(G)$  let  $T\mu(f) = \mu(\gamma(f \circ \alpha))$ . Then the mapping  $\mu \mapsto T\mu$  is an isometric  $*$ -isomorphism of  $M(F)$  onto  $M(G)$  (Lemma 2). In §3 we show that every norm decreasing isomorphism of  $M(F)$  onto  $M(G)$  is of the above form and consequently is an isometric  $*$ -isomorphism (Theorem 1). A number of other results follow from this, in particular Wendel's theorem on norm decreasing isomorphisms of group algebras.

Theorem 1 generalizes a result of B. E. Johnson [7] on isometric isomorphisms of measure algebras. In [7] Johnson used norm properties of measures in  $L^1(F)$  to show that each isometric isomorphism maps  $L^1(F)$  onto  $L^1(G)$  and consequently by applying Wendel's result it follows that each isometric isomorphism is of the form described above. Our generalization of Johnson's result has the advantage that Wendel's theorem is a consequence.

In §4 we use the results of §3 to prove a similar structure theorem for norm decreasing homomorphisms  $T$  of  $M(F)$  onto  $M(G)$  such that  $T(\mu * L^1(F)) = \{0\}$  implies  $T\mu = 0$ . (In this case the bicontinuous isomorphism of  $F$  onto  $G$  mentioned above becomes a continuous and open homomorphism of  $F$  onto  $G$ .)

The final section of the paper is concerned with bipositive homomorphisms of  $M(F)$  onto  $M(G)$ . Here we show that the hypothesis of norm decreasing may be dropped provided that the homomorphism maps the positive cone onto the positive cone.

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**2. Preliminaries.** Let  $(X, \tau)$  be a topological space, we write  $X_\tau$  in place of  $(X, \tau)$  and if  $Y \subseteq X$ ,  $Y_\tau$  is  $Y$  with the relative  $\tau$ -topology. If  $(x_j : j \in J)$  is a net in  $X$  which converges to  $x$  we shall write this as  $x_j \rightarrow x$ . The  $\tau$ -closure of a subset  $Y \subseteq X$  is written  $\text{Cl}_\tau Y$ .

Let  $G$  be a locally compact group,  $C(G)$  is the Banach space of bounded continuous complex valued functions on  $G$ ,  $K(G)$  the subspace of  $C(G)$  consisting of functions whose support is compact.  $C_0(G)$  may be defined as the closure of  $K(G)$  in  $C(G)$ .

As remarked in the introduction,  $M(G)$  is the dual of  $C_0(G)$ . The weak topology on  $M(G)$  is the  $\sigma(M(G), C_0(G))$ -topology, i.e. the coarsest topology on  $M(G)$  such that for each  $f$  in  $C_0(G)$  the mapping  $\mu \mapsto \mu(f)$  of  $M(G)$  into the complex field is continuous. We shall frequently write  $\sigma$  in place of  $\sigma(M(G), C_0(G))$ .

For  $x$  in  $G$ ,  $\varepsilon_x$  is the Dirac measure at  $x$ , i.e. the element of  $M(G)$  defined by  $\varepsilon_x(f) = f(x)$  for  $f \in C_0(G)$ .  $G^\varepsilon$  is the collection of all Dirac measures.

If  $\mu$  is any measure on  $G$ , then we shall use  $\mu$  for the corresponding integral, writing  $\mu(f) = \int f d\mu$  for any  $\mu$ -integrable function  $f$ .

The support of a measure  $\mu$ , written  $\text{Supp}(\mu)$  is the set of all  $x$  in  $G$  such that for each neighborhood  $U$  of  $x$ , there is an  $f$  in  $K(G)$  which vanishes outside  $U$  with  $\mu(f) \neq 0$ .

**PROPOSITION 1.** (i)  $G$  is isomorphic and homeomorphic to  $G^\varepsilon$ .

(ii) The mappings  $\mu \mapsto \mu * \lambda$ ,  $\mu \mapsto \lambda * \mu$  and  $\mu \mapsto \mu^*$  of  $M(G)$  into itself are  $\sigma(M(G), C_0(G))$ -continuous.

(iii) Let  $V$  be the linear span of  $G^\varepsilon$ . Then for any  $\mu \in M(G)$ ,

$$\mu \in \text{Cl}_\sigma \{ \lambda \in V : \|\lambda\| \leq \|\mu\| \text{ and } \text{Supp}(\lambda) \subseteq \text{Supp}(\mu) \}.$$

**Proof.** (i) It is straightforward to see that the mapping  $x \mapsto \varepsilon_x$  of  $G$  onto  $G^\varepsilon$  is a bicontinuous isomorphism.

(ii) For  $f \in C_0(G)$ , define  $\lambda^-(f)$  by  $\lambda^-(f)(x) = \lambda(xf)$  where  ${}_xf(y) = f(xy)$ . Then  $\lambda^-(f) \in C_0(G)$  [6, p. 264], and  $\mu * \lambda(f) = \mu(\lambda^-(f))$ . It follows that  $\mu \mapsto \mu * \lambda$  is  $\sigma(M(G), C_0(G))$ -continuous. Similarly the definition of  $\mu^*$  yields the continuity of the mapping  $\mu \mapsto \mu^*$  [6, p. 299]. To show that  $\mu \mapsto \lambda * \mu$  is continuous observe that this mapping can be written  $\mu \mapsto \mu^* \mapsto \mu^* * \lambda^* = (\lambda * \mu)^* \mapsto (\lambda * \mu)^{**} = \lambda * \mu$ .

The proof of (iii) is the same as the proof of Corollary 1 on p. 71 of [2].

**COROLLARY.** Given any  $\mu \in M(G)$  there is a net  $(\mu_j : j \in J)$  in

$$A = \{ \lambda \in V : \text{Supp}(\lambda) \subseteq \text{Supp}(\mu); \|\lambda\| \leq \|\mu\| \}$$

such that  $\mu_j \xrightarrow{\sigma} \mu$  and  $\|\mu\| = \lim \|\mu_j\|$ .

**Proof.**  $\mu_j \xrightarrow{\sigma} \mu$  implies  $\liminf \|\mu_j\| \geq \|\mu\|$  since the mapping  $\mu \mapsto \|\mu\|$  is lower semicontinuous in  $M(G)_\sigma$ .

Let  $L^1(G)$  be the subset of  $M(G)$  consisting of measures that are absolutely continuous with respect to the Haar measure on  $G$ . It is well known that  $L^1(G)$  is a

two-sided norm closed ideal in  $M(G)$ . Let  $B(L^1(G))$  be the algebra of all bounded linear operators which map  $L^1(G)$  into itself, taken with the topology of simple convergence (i.e. the coarsest topology such that for each  $\lambda \in L^1(G)$ , the mapping  $T \mapsto T\lambda$  is continuous.  $L^1(G)$  is taken with the norm topology). For  $\mu \in M(G)$  we define  $T_\mu \in B(L^1(G))$  by  $T_\mu \lambda = \mu * \lambda$ . The mapping  $\mu \mapsto T_\mu$  is one-one. The so-topology on  $M(G)$  is the coarsest topology such that this map is continuous. A theorem of Wendel [8] states that the image of  $M(G)$  under this mapping is closed in  $B(L^1(G))$ .

**PROPOSITION 2.** (i) *On norm bounded sets  $so \supset \sigma(M(G), C_0(G))$ .*

(ii)  $G_\sigma^\varepsilon = G_{so}^\varepsilon$ .

(iii) *The mappings  $\mu \mapsto \mu * \lambda$ ,  $\mu \mapsto \lambda * \mu$  are so-continuous.*

(iv) *Let  $V$  be the linear span of the Dirac measures. For any  $\mu \in M(G)$  we have  $\mu \in Cl_{so} \{ \lambda \in V : \text{Supp } (\lambda) \subseteq \text{Supp } (\mu) \text{ and } \|\lambda\| \leq \|\mu\| \}$ .*

(v) *Let  $\mu \in M(G)$  then  $\mu \in Cl_{so} \{ \lambda \in L^1(G) : \|\lambda\| \leq \|\mu\| \}$ .*

**Proof.** (i) follows from Lemma 1.1.1 of [4] and the fact that any  $f \in C_0(G)$  is left (and right) uniformly continuous [6, p. 185]. (ii) follows from (i) together with Proposition 1(i) and the fact that the mapping  $x \mapsto \varepsilon_x * \lambda$  is continuous for  $\lambda$  in  $L^1(G)$ . The proof of (iii) is an easy calculation. To prove (iv) use Lemmata 1.1.2 and 1.1.3 of [4]. The proof of (v) is an easy consequence of the facts that  $L^1(G)$  is an ideal in  $M(G)$  and  $L^1(G)$  contains an approximate unit of norm 1.

**COROLLARY.** *The mapping  $\lambda \mapsto \|\lambda\|$  is lower semicontinuous in the so-topology.*

The next proposition is the well-known relationship between the Dirac measures on  $G$  and the extreme points of the unit ball in  $M(G)$ .

**PROPOSITION 3.** *The extreme points of the unit ball of  $M(G)$  are precisely the measures  $\gamma \varepsilon_x$ ,  $|\gamma| = 1$ ,  $x \in G$ .*

**PROPOSITION 4.** *Let  $T$  be a norm bounded homomorphism of  $M(F)$  onto  $M(G)$  and suppose that for  $\mu$  in  $M(F)$ ,  $T(\mu * L^1(F)) = \{0\}$  implies  $T\mu = 0$ . Then  $T$  restricted to norm bounded sets is continuous as a mapping of  $M(F)_{so}$  onto  $M(G)_\sigma$ .*

**Proof.** Let  $(e_j : j \in J)$  be an approximate unit in  $L^1(F)$ . By Lemma 4.1.2 of Greenleaf [4],  $Te_j \xrightarrow{\sigma} \varepsilon$  where  $\varepsilon$  is an idempotent such that  $\varepsilon * \mu = \mu * \varepsilon = \mu$  for all  $\mu$  in  $Cl_\sigma T(L^1(F))$ . Since  $T$  is onto and  $T(\mu * L^1(F)) = \{0\}$  implies  $T\mu = 0$ , it follows that  $\varepsilon$  is the unit of  $M(G)$ . Let  $(\mu_j : j \in J)$  be a norm bounded net in  $M(F)$  and suppose that  $\mu_j \xrightarrow{so} \mu \in M(F)$ . For any  $\lambda$  in  $L^1(F)$  we have  $\mu_j * \lambda \rightarrow \mu * \lambda$  so that  $T\mu_j * T\lambda \rightarrow T\mu * T\lambda$ . Since  $(T\mu_j : j \in J)$  is norm bounded and the unit ball of  $M(G)$  is  $\sigma(M(G), C_0(G))$ -compact it follows that there is a  $\mu' \in M(G)$  and a subnet  $(T\mu_{j(i)})$  such that  $T\mu_{j(i)} \xrightarrow{\sigma} \mu'$ . Consequently for any  $\lambda$  in  $L^1(F)$  we have  $T\mu * T\lambda = \mu' * T\lambda$ , since multiplication is separately continuous in  $M(G)_\sigma$ . Thus  $T\mu * Te_j = \mu' * Te_j$ , and since  $Te_j \xrightarrow{\sigma} \varepsilon$ , we have  $T\mu = \varepsilon * T\mu = \mu'$ . This completes the proof.

LEMMA 1. Let  $F, G$  be locally compact groups and let  $T$  be a norm decreasing homomorphism of  $M(F)$  onto  $M(G)$ . Then for each  $x \in F$  there is a  $y \in G$ , and a complex number  $\gamma$ ,  $|\gamma|=1$  such that  $T\varepsilon_x = \gamma\varepsilon_y$ .

**Proof.** Let  $L_x$  be the operator on  $M(G)$  defined by  $L_x\mu = T\varepsilon_x * \mu$ . Since  $T$  is a norm decreasing homomorphism of  $M(F)$  onto  $M(G)$  it follows that  $L_x^{-1}$  exists;  $L_x^{-1} = L_x^{-1}$  and  $\|L_x\| \leq 1$ ,  $\|L_x^{-1}\| \leq 1$ . Let  $\mu, \lambda \in M(G)$ ,  $\|\mu\| \leq 1$ ,  $\|\lambda\| \leq 1$ , and suppose that  $T\varepsilon_x = a\mu + (1-a)\lambda$  for  $0 < a < 1$ . Then  $\varepsilon_e = aL_x^{-1}\mu + (1-a)L_x^{-1}\lambda$ , and since  $\varepsilon_e$  is an extreme point of the unit ball, we have  $\mu = \lambda$ . Thus  $T\varepsilon_x$  is an extreme point of the unit ball of  $M(G)$  so now the lemma follows from Proposition 3.

### 3. Norm decreasing isomorphisms of measure algebras.

LEMMA 2. Let  $F$  and  $G$  be locally compact groups,  $\alpha$  a bicontinuous isomorphism of  $F$  onto  $G$  and  $\gamma$  a character on  $F$ . Let  $T$  be the mapping of  $M(F)$  into  $M(G)$  defined by

$$T\mu(f) = \mu(\gamma(f \circ \alpha)), \quad \mu \in M(F), f \in C_0(G).$$

Then  $T$  is an isometric  $*$ -isomorphism of  $M(F)$  onto  $M(G)$ ; and  $T$  is a bicontinuous mapping of  $M(F)_\sigma$  onto  $M(G)_\sigma$ .

**Proof.** Consider the mapping  $f \mapsto \gamma(f \circ \alpha)$  of  $C_0(G)$  into  $C_0(F)$ . It is easily seen that this mapping is an isometry of  $C_0(G)$  onto  $C_0(F)$  and that  $T$  is its adjoint. Consequently  $T$  is an isometry of  $M(F)$  onto  $M(G)$  and is a bicontinuous mapping of  $M(F)_\sigma$  onto  $M(G)_\sigma$ . That  $T(\mu * \lambda) = T\mu * T\lambda$  and  $(T\mu)^* = T\mu^*$  hold may be shown by calculations.

THEOREM 1. Let  $F$  and  $G$  be locally compact groups and let  $T$  be a norm decreasing isomorphism of  $M(F)$  onto  $M(G)$ . Then there is a bicontinuous isomorphism  $\alpha$  of  $F$  onto  $G$ , and a character  $\gamma$  on  $F$  such that

$$T\mu(f) = \mu(\gamma(f \circ \alpha)), \quad \mu \in M(F), f \in C_0(G).$$

**Proof.** For  $x \in F$  we have by Lemma 1, that there is a complex number  $\gamma(x)$  with  $|\gamma(x)|=1$ , and an element  $\alpha(x)$  of  $G$  such that  $T\varepsilon_x = \gamma(x)\varepsilon_{\alpha(x)}$ . Consider the mappings

$$\gamma: x \mapsto \gamma(x) \quad \text{and} \quad \alpha: x \mapsto \alpha(x).$$

We first show that  $\alpha$  is a homomorphism of  $F$  into  $G$ , and  $\gamma$  is a homomorphism of  $F$  into the complex numbers of absolute value 1. Clearly  $T\varepsilon_{xy} = \gamma(xy)\varepsilon_{\alpha(xy)}$ , and

$$T\varepsilon_x * T\varepsilon_y = \gamma(x)\gamma(y)\varepsilon_{\alpha(x)} * \varepsilon_{\alpha(y)} = \gamma(x)\gamma(y)\varepsilon_{\alpha(x)\alpha(y)}.$$

Since  $T\varepsilon_x * T\varepsilon_y = T(\varepsilon_x * \varepsilon_y) = T\varepsilon_{xy}$ , we have

$$\gamma(xy)\varepsilon_{\alpha(xy)} = \gamma(x)\gamma(y)\varepsilon_{\alpha(x)\alpha(y)};$$

and since the Dirac measures are pairwise linearly independent, we have  $\gamma(xy) = \gamma(x)\gamma(y)$ , and  $\alpha(xy) = \alpha(x)\alpha(y)$ . Thus  $\alpha$  is a homomorphism of  $F$  into  $G$ , and  $\gamma$  is a homomorphism of  $F$  into the complex numbers.

We now show that  $\gamma$  is continuous. For any nonnegative  $f$  in  $C_0(G)$  we have  $|T\epsilon_x(f)| = \epsilon_{\alpha(x)}(f)$  consequently  $\gamma(x)|T\epsilon_x(f)| = T\epsilon_x(f)$ . Thus to show the continuity of  $\gamma$  it suffices to show that the mapping  $x \mapsto T\epsilon_x(f)$  is continuous at  $e$  in  $F$  where  $f$  is a nonnegative function in  $C_0(G)$  such that  $T\epsilon_e(f) \neq 0$ . By the definition of the weak topology the mapping  $T\epsilon_x \mapsto T\epsilon_x(f)$  is continuous from  $M(G)_\sigma$  into the complex numbers. The mapping  $x \mapsto \epsilon_x$  is continuous from  $F$  to  $F_{s_0}^\epsilon$  by Propositions 1 and 2 and the mapping  $\epsilon_x \mapsto T\epsilon_x$  is continuous from  $F_{s_0}^\epsilon$  to  $M(G)_\sigma$  by Proposition 4. It follows that  $\gamma$  is continuous and thus  $\gamma$  is a character.

The continuity of  $\alpha$  follows by considering the mappings

$$x \mapsto \epsilon_x \mapsto T\epsilon_x = \gamma(x)\epsilon_{\alpha(x)} \mapsto \epsilon_{\alpha(x)} \mapsto \alpha(x).$$

The only mapping we have to check is the mapping  $\gamma(x)\epsilon_{\alpha(x)} \mapsto \epsilon_{\alpha(x)}$ , of a subset of  $M(G)_\sigma$  into  $M(G)_\sigma$ . But since this mapping is multiplication by the continuous character  $x \mapsto (\gamma(x))^{-1}$ , it is continuous. Thus  $\alpha$  is continuous since it may be written as a composite of continuous mappings. Moreover since each of the above mappings is one to one we have that  $\alpha$  is a continuous one to one homomorphism. Now consider  $T^{-1}$ . By the open mapping theorem  $T^{-1}$  is a bounded isomorphism of  $M(G)$  onto  $M(F)$ . Thus Proposition 4 applies and we have that  $\epsilon_{\alpha(x)} \mapsto T^{-1}\epsilon_{\alpha(x)} = (\gamma(x))^{-1}\epsilon_x$  is continuous from  $G_{s_0}^\epsilon$  into  $M(F)_\sigma$ . The continuity of  $\alpha^{-1}$  restricted to  $\alpha(F)$  now follows by considering the mappings

$$\alpha(x) \mapsto \epsilon_{\alpha(x)} \mapsto T^{-1}\epsilon_{\alpha(x)} = (\gamma(x))^{-1}\epsilon_x \mapsto \epsilon_x \mapsto x.$$

Thus  $F$  is homeomorphic to  $\alpha(F)$  and since a locally compact group is complete  $\alpha(F)$  is complete and therefore closed.

Now suppose  $\alpha$  is not onto; then there is a  $y$  in  $G \setminus \alpha(F)$  and a compact neighborhood  $V$  of  $y$  such that  $V \cap \alpha(F) = \emptyset$  because  $\alpha(F)$  is closed. Since  $T^{-1}\epsilon_y$  is in  $M(F)$ , by Proposition 2 there is a net  $(\mu_j : j \in J)$  such that

$$\mu_j = \sum_{i=1}^{n_j} b_{i,j} \epsilon_{x_{i,j}}, \quad x_{i,j} \in F, b_{i,j} \text{ complex,}$$

$$\|\mu_j\| \leq \|T^{-1}\epsilon_y\|$$

and  $\mu_j \xrightarrow{s_0} T^{-1}\epsilon_y$ . Thus by Proposition 4,  $T\mu_j \xrightarrow{\sigma} \epsilon_y$ . Note that  $T\epsilon_x = \gamma(x)\epsilon_{\alpha(x)}$ , thus

$$T\mu_j = \sum_{i=1}^{n_j} b_{i,j} \gamma(x_{i,j}) \epsilon_{\alpha(x_{i,j})}.$$

Since  $G$  is locally compact there is a function  $f$  in  $C_0(G)$  such that  $f(y)=1$  and  $f(G \setminus V)=0$  and  $0 \leq f(x) \leq 1$  for all  $x \in G$ . Since  $T\mu_j \xrightarrow{\sigma} \epsilon_y$  we have

$$\sum_{i=1}^{n_j} b_{i,j} \gamma(x_{i,j}) \epsilon_{\alpha(x_{i,j})}(f) \rightarrow f(y).$$

But since  $\alpha(x_{i,j}) \in \alpha(F) \cap (G \setminus V)$ ,  $\epsilon_{\alpha(x_{i,j})}(f) = f(\alpha(x_{i,j})) = 0$ , so that  $f(y)=0$ , a contradiction. Thus  $\alpha$  is onto.

All that remains now is to establish the formula  $(T\mu)(f) = \mu(\gamma(f \circ \alpha))$ .

Let  $T_1$  be the mapping defined by

$$(T_1\mu)(f) = \mu(\gamma(f \circ \alpha)), \quad \mu \in M(F), f \in C_0(G).$$

By Lemma 2 we have that  $T_1$  is an isomorphism and isometry from  $M(F)$  onto  $M(G)$ . Hence in view of Proposition 4,  $T_1$  is continuous on norm bounded sets from  $M(F)_{so}$  onto  $M(G)_o$ . Now observe that

$$T_1\varepsilon_x(f) = \varepsilon_x(\gamma(f \circ \alpha)) = \gamma(x)f(\alpha(x)) = \gamma(x)\varepsilon_{\alpha(x)}(f) = T\varepsilon_x(f)$$

Thus  $T$  and  $T_1$  coincide on  $F^\varepsilon$ , and by Proposition 2 each  $\mu \in M(F)$  is a *so*-adherence point of a norm bounded set of linear combinations of Dirac measures so we have  $T = T_1$ . This completes the proof.

**COROLLARY 1.** *Every norm decreasing isomorphism of  $M(F)$  onto  $M(G)$  is an isometric \*-isomorphism.*

**Proof.** This follows from Lemma 2 and the above theorem.

**COROLLARY 2.** *Let  $T$  be a norm decreasing isomorphism of  $M(F)$  onto  $M(G)$ ; then  $T$  is a bicontinuous mapping of  $M(F)_o$  onto  $M(G)_o$ .*

**Proof.** This follows from Lemma 2 and the above theorem.

**COROLLARY 3.** *Each norm decreasing isomorphism of  $M(F)$  onto  $M(G)$  maps  $L^1(F)$  onto  $L^1(G)$ .*

**Proof.** This follows from the formula.

**COROLLARY 4.**  *$L^1(G)$  is invariant under norm decreasing automorphisms of  $M(G)$ .*

The following example shows that a \*-isomorphism of  $M(F)$  onto  $M(G)$  need not be norm decreasing. Let  $F$  and  $G$  be finite abelian groups of order  $n$ , and suppose that  $F$  and  $G$  are not isomorphic. Let  $\gamma_1, \gamma_2, \dots, \gamma_n$  be the  $n$  characters of  $F$  and define functions  $f_i, i = 1, 2, \dots, n$  on  $F$  by

$$f_i(x) = (1/n)\gamma_i(x), \quad x \in F, i = 1, 2, \dots, n.$$

Using the orthogonality relations for characters on abelian groups it follows that  $f_1, f_2, \dots, f_n$  are orthogonal idempotents which span  $M(F)$ . Clearly  $f_i^* = f_i$ . Let  $g_1, g_2, \dots, g_n$  be a basis for  $M(G)$  defined in the analogous manner. Define  $T$  by  $Tf_i = g_i, i = 1, 2, \dots, n$  and extend  $T$  to a linear mapping of  $M(F)$  into  $M(G)$ . It is easily seen that  $T$  is a \*-isomorphism of  $M(F)$  onto  $M(G)$ .  $T$  cannot be norm decreasing if  $F$  and  $G$  are not isomorphic since this would violate Theorem 1.

As a consequence of Theorem 1 we shall derive a theorem due to Wendel [8] on isomorphisms of  $L^1(F)$  onto  $L^1(G)$ . First we shall need the following lemma.

**LEMMA 3.** *Let  $T$  be a bounded isomorphism of  $L^1(F)$  onto  $L^1(G)$ . Then there is a unique bounded isomorphism  $T^\#$  of  $M(F)$  onto  $M(G)$  which extends  $T$ . Moreover,  $\|T^\#\| = \|T\|$ .*

**Proof.** Clearly  $T$  is continuous as a mapping of  $L^1(F)_{so}$  onto  $L^1(G)_{so}$ . Since  $M(G)_{so}$  is quasi-complete (see the remark preceding Proposition 2) and since each  $\mu$  in  $M(F)$  is a  $so$ -adherence point of a bounded set in  $L^1(F)$  (Proposition 2),  $T$  has a unique extension  $T^\#$  to a continuous linear mapping of  $M(F)_{so}$  into  $M(G)_{so}$  [1, Chapitre III, §2, No. 5].

To show that  $T^\#(\mu * \lambda) = T^\#\mu * T^\#\lambda$ ,  $\mu, \lambda$  in  $M(F)$ , let  $(\mu_j : j \in J)$  and  $(\lambda_k : k \in K)$  be nets in  $L^1(F)$  such that  $\mu_j \xrightarrow{so} \mu$  and  $\lambda_k \xrightarrow{so} \lambda$ . Then since multiplication is separately continuous in  $M(F)_{so}$  (Proposition 2) we have

$$\mu * \lambda = \lim_j \left( \lim_k \mu_j * \lambda_k \right).$$

Therefore the  $so$ -continuity of  $T^\#$  implies that

$$T^\#(\mu * \lambda) = \lim_j \left( \lim_k T(\mu_j * \lambda_k) \right).$$

Since  $T^\# = T$  on  $L^1(F)$  we have  $T^\#(\mu_j * \lambda_k) = T\mu_j * T\lambda_k$ . Now using the fact that multiplication is separately continuous in  $M(G)_{so}$  we have

$$\lim_j \left( \lim_k T\mu_j * T\lambda_k \right) = T^\#\mu * T^\#\lambda.$$

Combining the above we have  $T^\#(\mu * \lambda) = T^\#\mu * T^\#\lambda$ .

To show that  $T^\#$  is one-one, let  $\lambda, \mu \in M(F)$  and suppose  $T^\#\mu = T^\#\lambda$ . If  $\lambda \neq \mu$  then there is a  $\nu$  in  $L^1(F)$  such that  $\lambda * \nu \neq \mu * \nu$ . Then  $T(\lambda * \nu) = T^\#\lambda * T\nu = T^\#\mu * T\nu = T(\mu * \nu)$  which contradicts the assertion that  $T$  is an isomorphism because  $\lambda * \nu$  and  $\mu * \nu$  are in  $L^1(F)$ . Therefore  $T^\#$  is one-one.

We now show that  $T^\#$  is onto. Let  $\mu' \in M(G)$ , by Proposition 2 there is a net  $(\mu'_j : j \in J)$  in  $L^1(G)$  such that  $\mu'_j \xrightarrow{so} \mu'$  and  $\|\mu'_j\| \leq \|\mu'\|$ . By the open mapping theorem  $T^{-1}$  is bounded, so that  $(T^{-1}\mu'_j : j \in J)$  is a bounded Cauchy net in  $L^1(F)_{so}$ . Since  $M(F)_{so}$  is quasi-complete there is a  $\mu$  in  $M(F)$  such that  $T^{-1}\mu'_j \xrightarrow{so} \mu$ . Then  $T(T^{-1}\mu'_j) \xrightarrow{so} T\mu$  so that  $T^\#\mu = \mu'$  and  $T^\#$  is onto.

We next show  $\|T^\#\| = \|T\|$ . Clearly  $\|T\| \leq \|T^\#\|$ . To show the reverse inequality let  $\mu \in M(F)$  be given. By Proposition 2 there is a net  $(\mu_j : j \in J)$  in  $L^1(F)$  such that  $\mu_j \xrightarrow{so} \mu$  and  $\|\mu_j\| \leq \|\mu\|$  so that  $T\mu_j \xrightarrow{so} T^\#\mu$ . Since the mapping  $\lambda \mapsto \|\lambda\|$  is lower semicontinuous in the  $so$ -topology (Corollary 1 to Proposition 2), we have  $\|T^\#\mu\| \leq \liminf \|T\mu_j\| \leq \liminf \|T\| \|\mu_j\| \leq \|T\| \|\mu\|$ . Therefore  $\|T^\#\| \leq \|T\|$  and hence  $\|T^\#\| = \|T\|$ .

The uniqueness of  $T^\#$  is an easy consequence of Propositions 2 and 4.

**THEOREM 2 (WENDEL).** *Let  $T$  be a norm decreasing isomorphism of  $L^1(F)$  onto  $L^1(G)$ . Then there is a bicontinuous isomorphism  $\alpha$  of  $F$  onto  $T$ , a character  $\gamma$  on  $G$  such that*

$$T\lambda(f) = \lambda(\gamma(f \circ \alpha)) \quad \lambda \in L^1(F), f \in C_0(G).$$

**Proof.** This follows easily from Theorem 1 in virtue of Lemma 3.

**COROLLARY.** *Each norm decreasing isomorphism of  $L^1(F)$  onto  $L^1(G)$  is an isometric  $*$ -isomorphism.*

**4. Norm decreasing homomorphisms of measure algebras.** In this section we give the structure of all norm decreasing homomorphisms  $T$  of  $M(F)$  onto  $M(G)$  such that for  $\mu \in M(F)$ ,  $T(\mu * L^1(F)) = \{0\}$  implies  $T\mu = 0$ .

**LEMMA 4.** *Let  $F$  and  $G$  be locally compact groups,  $\alpha$  a continuous and open homomorphism of  $F$  onto  $G$ , and  $\gamma$  a character on  $F$ . Let  $T$  be the mapping of  $M(F)$  into  $M(G)$  defined by*

$$T\mu(f) = \mu(\gamma(f \circ \alpha)) \quad \mu \in M(F), f \in C_0(G).$$

*Then  $T$  is a norm decreasing \*-homomorphism of  $M(F)$  onto  $M(G)$ ;  $T$  is continuous as a mapping of  $M(F)_{so}$  onto  $M(G)_{so}$  and  $T(L^1(F)) = L^1(G)$ .*

**Proof.** Let  $e$  be the unit of  $G$  and let  $F_0 = \{x : \alpha(x) = e\}$ . Then  $F_0$  is a closed normal subgroup of  $F$ . Let  $\pi: F \rightarrow F/F_0$  be the canonical mapping then there is a bicontinuous isomorphism  $\beta: F/F_0 \rightarrow G$  such that  $\alpha = \beta \circ \pi$ . Consider the following mappings:

- (1)  $\gamma: M(F) \rightarrow M(F)$  defined by  $\gamma\mu(f) = \mu(\gamma f)$   $\mu \in M(F), f \in C_0(F)$ ,
- (2)  $\pi^{**}: M(F) \rightarrow M(F/F_0)$  defined by  $\pi^{**}\mu(f) = \mu(f \circ \pi)$   $\mu \in M(F), f \in C_0(F/F_0)$ ,
- (3)  $\beta^{**}: M(F/F_0) \rightarrow M(G)$  defined by  $\beta^{**}\mu(f) = \mu(f \circ \beta)$   $\mu \in M(F/F_0), f \in C_0(G)$ .

By Lemma 2, (1) and (3) are isometric isomorphisms. By Lemma 5.1.8 of Greenleaf [4],  $\pi^{**}$  is a norm decreasing homomorphism of  $M(F/F_0)$  onto  $M(G)$ . Observe that  $T = \beta^{**}\pi^{**}\gamma$  so that  $T$  is a norm decreasing homomorphism of  $M(F)$  onto  $M(G)$ . That  $T$  is a \*-homomorphism follows by a calculation. To complete the proof it suffices to show that  $T(L^1(F)) = L^1(G)$ . In the proof of Theorem 5.2.1 of [4] Greenleaf shows that  $\pi^{**}(L^1(F)) = L^1(F/F_0)$  consequently by Corollary 3 to Theorem 1,  $T(L^1(F)) = L^1(G)$ .

The next lemma is an observation of Greenleaf's [4] and [5].

**LEMMA 5.** *Let  $F$  and  $G$  be locally compact groups;  $\alpha$  a continuous homomorphism of  $F$  into  $G$  and  $\gamma$  a character of  $F$ . Then the mapping  $T$  defined by  $T\mu(f) = \mu(\gamma(f \circ \alpha))$  is a norm decreasing homomorphism of  $M(F)$  into  $M(G)$  which is continuous on norm bounded sets as a mapping of  $M(F)_{so}$  into  $M(G)_\sigma$ . If  $\alpha$  is a monomorphism then so is  $T$ .*

**Proof.** It is straightforward to verify that  $T$  is a norm decreasing homomorphism of  $M(F)$  into  $M(G)$ . The stated continuity property of  $T$  follows from Lemma 1.1.1 of Greenleaf [4] and the fact that for  $f \in C_0(G)$ ,  $f \circ \alpha$  is left (and right) uniformly continuous on  $F$ . If  $\alpha$  is a monomorphism then to show that  $T$  is a monomorphism it suffices to show that the mapping  $\alpha^{**}$  defined by  $\alpha^{**}\mu(f) = \mu(f \circ \alpha)$ ,  $\mu \in M(F)$ ,  $f \in C_0(G)$  is a monomorphism. To show this let  $K \subseteq F$  be any compact subset, then  $\mu(K) = \mu(\alpha^{-1}\alpha(K)) = \alpha^{**}\mu(\alpha(K))$  since  $\alpha$  is a monomorphism. It follows that  $\alpha^{**}\mu = 0$  implies  $\mu = 0$  [6, p. 175]. This completes the proof.



**THEOREM 3.** *Let  $T$  be a norm decreasing homomorphism of  $M(F)$  onto  $M(G)$ . If  $T(\mu * L^1(F)) = \{0\}$  implies  $T\mu = 0$ , then there is a character  $\gamma$  on  $F$  and a continuous and open homomorphism  $\alpha$  of  $F$  onto  $G$  such that*

$$T\mu(f) = \mu(\gamma(f \circ \alpha)) \quad \text{for } \mu \in M(F), f \in C_0(G).$$

**Proof.** For  $x \in F$  we have by Lemma 1, a complex number  $\gamma(x)$ ,  $|\gamma(x)| = 1$  and an  $\alpha(x) \in G$  such that  $T\varepsilon_x = \gamma(x)\varepsilon_{\alpha(x)}$ . It follows as in the proof of Theorem 1 that  $\alpha$  is a continuous homomorphism of  $F$  into  $G$ ;  $\gamma$  is a character of  $F$  and that  $T\mu(f) = \mu(\gamma(f \circ \alpha))$  (in virtue of Lemma 5). To complete the proof it remains to show that  $\alpha$  is open and onto. Let  $e$  be the unit of  $G$  and let  $F_0 = \{x : \alpha(x) = e\}$ . Let  $\pi: F \rightarrow F/F_0$  be the canonical homomorphism and define  $T_1: M(F) \rightarrow M(F/F_0)$  by  $T_1\mu(f) = \mu(\gamma(f \circ \pi))$ . By Lemma 4,  $T_1$  is a norm decreasing homomorphism which is onto. Let  $\mu \in \ker T_1$ , then  $\mu(\gamma(f \circ \pi)) = 0$  for all  $f$  in  $C_0(F/F_0)$  and it follows from Lemma 5.2.2 of Greenleaf [4], that  $\mu(\gamma(f \circ \pi)) = 0$  for all  $f \in C(F/F_0)$ . Let  $\beta: F/F_0 \rightarrow G$  be such that  $\alpha = \beta \circ \pi$ . For  $f \in C_0(G)$ ,  $f \circ \beta \in C(F/F_0)$  so that  $\mu(\gamma(f \circ \beta \circ \pi)) = 0$ . Thus we have that  $\ker T_1 \subseteq \ker T$  so we may define a homomorphism  $T_2: M(F/F_0) \rightarrow M(G)$  by  $T_2(T_1\mu) = T\mu$ . To show that  $T_2$  is an isomorphism we need only show that  $T_2$  is a monomorphism since  $T_2$  is clearly onto. Define  $T': M(F/F_0) \rightarrow M(G)$  by  $T'\mu(f) = \mu(f \circ \beta)$ , then  $T'T_1\mu(f) = T_1\mu(f \circ \beta) = \mu(\gamma(f \circ \beta \circ \pi)) = T\mu(f) = T_2T_1\mu(f)$  for all  $\mu$  in  $M(F)$ . Since  $T$  and  $T_1$  are onto,  $T_2 = T'$ . Now since  $\beta$  is a continuous monomorphism,  $T_2$  is a norm decreasing monomorphism (Lemma 5).

If we now apply Theorem 1 to  $T_2$  we have that there is a character  $\gamma'$  on  $F/F_0$  and a bicontinuous isomorphism  $\alpha'$  of  $F/F_0$  onto  $G$  such that

$$T_2(T_1\mu)(f) = T_1\mu(\gamma'(f \circ \alpha')), \quad \mu \in M(F) \text{ and } f \in C_0(G).$$

Thus using the definition of  $T_1$  we have

$$\mu(\gamma(\gamma' \circ \pi)(f \circ \alpha' \circ \pi)) = \mu(\gamma(f \circ \alpha)), \quad \mu \in M(F) \text{ and } f \in C_0(G).$$

It follows that  $\alpha' \circ \pi = \alpha$  and consequently  $\alpha$  is open and onto.

**COROLLARY 1.** *Let  $T$  be a norm decreasing homomorphism of  $M(F)$  onto  $M(G)$ . If  $T(\mu * L^1(F)) = \{0\}$  implies  $T\mu = 0$ , then  $T(L^1(F)) = L^1(G)$  and consequently  $T$  is continuous as a mapping of  $M(F)_{so}$  onto  $M(G)_{so}$ .*

**Proof.** This follows from the theorem and Lemma 4.

**COROLLARY 2.** *Each norm decreasing homomorphism of  $M(F)$  onto  $M(G)$  satisfying the hypothesis of Theorem 3 is a  $*$ -homomorphism.*

**REMARK.** There are homomorphisms  $T$  which are onto but which do not satisfy the hypothesis of the theorem as the following example shows.

Let  $F$  be a nondiscrete locally compact group and let  $G = \{e\}$  be the group consisting of only one element. Define  $T: M(F) \rightarrow M(G)$  by  $T\mu = \sum_{x \in F} \mu(\{x\})\varepsilon_e$ . Then  $T$

is a norm decreasing homomorphism which is onto and  $T$  annihilates  $L^1(F)$ . Corollary 1 to the theorem shows that the conclusion of Theorem 2 cannot hold for  $T$ .

### 5. Bipositive homomorphisms of measure algebras.

**DEFINITION.** A homomorphism  $T$  of  $M(F)$  onto  $M(G)$  is said to be bipositive if  $T$  maps the set of positive measures onto the set of positive measures.

**THEOREM 4.** *Let  $T$  be a bipositive homomorphism of  $M(F)$  onto  $M(G)$ . If  $T(\mu * L^1(F)) = \{0\}$  implies  $T\mu = 0$  then there is a continuous and open homomorphism  $\alpha$  of  $F$  onto  $G$  such that*

$$T\mu(f) = \mu(f \circ \alpha), \quad \mu \in M(F), f \in C_0(G).$$

*If  $T$  is an isomorphism then so is  $\alpha$ .*

**Proof.** Note that a positive measure  $\mu$  on  $G$  is a real multiple of a Dirac measure if and only if each  $\lambda \in M(G)$  for which  $\mu - \lambda$  is a positive measure is a real multiple of  $\mu$ . Since this property is clearly preserved under bipositive maps we have that for each  $x \in F$ , there is a real number  $\gamma(x)$  and an  $\alpha(x) \in G$  such that  $T\epsilon_x = \gamma(x)\epsilon_{\alpha(x)}$ . It is readily verified that the mapping  $x \mapsto \gamma(x)$  is a homomorphism of  $G$  into the positive reals. Since  $T$  is positive,  $T$  is bounded, thus there is a constant  $M$  such that  $\|T\mu\| \leq M\|\mu\|$ , for all  $\mu \in M(F)$ . It follows that  $\{\gamma(x) : x \in G\}$  is a bounded subgroup of the positive reals. Thus we must have  $\gamma(x) = 1$  for all  $x \in F$ . Applying Proposition 4 we see that  $T\mu(f) = \mu(f \circ \alpha)$  for  $\mu \in M(F), f \in C_0(G)$ . By Lemma 5,  $T$  is a norm decreasing homomorphism and Theorems 1 and 3 now apply.

**COROLLARY.** *Let  $T$  be a bipositive homomorphism of  $M(F)$  onto  $M(G)$ . If  $T(\mu * L^1(F)) = \{0\}$  implies  $T\mu = 0$  then  $T$  is a norm decreasing \*-homomorphism.*

**REMARK.** The example given after Corollary 2 to Theorem 3 shows that there are bipositive homomorphisms which do not satisfy the hypothesis of the theorem.

The author has recently learned that G. Gaudry has proven Theorem 4 for the case when  $T$  is a bipositive isomorphism (to appear, *Canad. J. Math.*).

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